Perturbation theory around two-dimensional critical systems through holomorphic decomposition

1990 J. Phys. A: Math. Gen. 23 2971

(http://iopscience.iop.org/0305-4470/23/13/033)

J. Phys. A: Math. Gen. 23 (1990) 2971-2986. Printed in the UK

Perturbation theory around two-dimensional critical systems through holomorphic decomposition

F Constantinescu† and R Flume‡

† Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, Robert-Mayer-Strasse 10, D-6000 Frankfurt, Federal Republic of Germany

‡ Physikalisches Institut, Universität Bonn, Nussallee 12, D-5300 Bonn 1, Federal Republic of Germany

Received 28 December 1989

Abstract. We consider perturbations by relevant interactions of two-dimensional conformally invariant rational field theories. The meromorphic structure—with respect to the scaling dimensions of the perturbing interactions—of the correlation functions is made explicit through a successive application of Stoke's theorem. The resulting decomposition of the amplitudes into holomorphic and antiholomorphic factors yields a representation of the meromorphic structure in terms of the basic data of the rational field theory, which are scaling dimensions, operator product coefficients and braid matrices. We exemplify the general deduction by a concrete calculation to the third order in the coupling constant of the perturbing interaction.

1. Introduction

The benefits of conformal invariance in any number of spacetime dimensions for the simplification of quantum field theories have been well known for a long time [1]. The even greater benefits of the infinite-dimensional conformal symmetry in two spacetime dimensions, first realised in the pioneering paper of Belavin *et al* [2], led to the discovery of several series of (by now called) rational conformally invariant field theories (RCFT) [3] which by definition are theories with a finite number of primary operators (that is, primary with respect to the conformal symmetry or an extension thereof). We want to investigate in this paper perturbation theory around these RCFT, with conformally non-invariant additions to the action of the form

$$\Delta S = \boldsymbol{g}\cdot\int \mathrm{d}^2x\, \boldsymbol{\psi}(x) \qquad \boldsymbol{g}=(g_1,\ldots,g_k) \qquad \boldsymbol{\psi}=(\psi_1,\ldots,\psi_k) \tag{1}$$

where $\boldsymbol{g}$ denotes an ensemble of coupling constants and $\boldsymbol{\psi}$ stands for a collection of primary operators of the RCFT we start from with relevant scaling dimensions $d_{\psi_i}<2$ but rather near to marginality. We hope to mediate the insight to the reader that the striking structural simplicity of RCFT has its repercussions in the here investigated perturbation theory (PT) in so far as the latter is much easier to control than most of the conventional PT around Gaussian fixed points.

The kind of PT around RCFT we are aiming at was first considered by Zamolodchikov [4], by Ludwig [5] and Ludwig and Cardy [6] (see also [7]). Those authors drew conclusions from calculations up to second order in the coupling. We intend to describe here in some detail the general structure of the PT. We will derive a representation of the perturbative corrections to correlation functions, the main virtue of which is that it gives a concise and explicit determination of the meromorphic behaviour in the scaling dimensions of the operators coming into focus. We will perform towards the end of the paper a concrete perturbative calculation to third order in the coupling which displays the essential features showing up at all orders of PT.

There are indications [8] that some of the models, to which our methods apply, may be integrable systems (away from the critical point). Considerations pertinent to integrability will not play any role in the following.

2. Rational conformal field theory

We recall some basic facts concerning the structure of RCFTs. We give by no means a complete account, for which we refer to [9-12], but confine ourselves to collect some material in order to make the succeeding argument reasonably self-contained.

The field content of the RCFT to be considered is given by:

(i) finitely many chiral fields (among them the z-z component T_{zz} of the energy momentum tensor) generating a chiral symmetry algebra E which may be the Virasoro algebra or one of its extensions;

(ii) an anti-chiral symmetry algebra of field $\bar{E}$ isomorphic to E (we assume parity invariance);

(iii) a finite number, say N, of physical fields, denoted by $\Phi_1, \ldots, \Phi_N$, being primary with respect to the symmetry algebras E and $\bar{E}$;

(iv) descendent fields which arise through action of the Laurent components of the fields in E and $\bar{E}$ on the $\Phi_{\mathcal{I}}$ and the unit operator 1.

The correlation functions of the physical primary fields (as well as those of their descendants) decompose into sums of terms each of which factorises into a holomorphic (chiral) and an antiholomorphic (anti-chiral) piece. The pieces are called 'conformal blocks'. A convenient description of the block structure is achieved through the introduction of chiral vertex operators [9]. We assume for simplicity that the physical primary operators are scalars under spacetime rotations. They have as such isomorphic transformation properties vis à vis E and $\bar{E}$. The states $\Phi_{\mathcal{I}}(0)|0\rangle \equiv |\mathcal{I}\rangle$, with $|0\rangle$ denoting the ground state of our RCFT, are lowest-weight states of representations of E and $\bar{E}$ which are built up through the action of the two algebras on $|\mathcal{I}\rangle$. $\Phi_{\mathcal{I}}$ may be thought of as being composed of a chiral field φ_i and an anti-chiral field $\bar{\varphi}_i$, with φ_i and $\overline{\varphi_i}$ mediating the corresponding lowest-weight representations of E and $\bar{E}$, respectively. We introduce the notations

$$\begin{aligned} &\varphi_i(0)|0\rangle \equiv |i\rangle \qquad \bar{\varphi}_i(0)|0\rangle \equiv \bar{i}\rangle \\ &|i\rangle \otimes |\bar{i}\rangle \equiv |\mathcal{I}\rangle \\ &H_i = E|i\rangle,\ \bar{H}_i = \bar{E}|\bar{i}\rangle. \end{aligned} \tag{2}$$

The space H of physical states is given by the sum

$$\begin{aligned} &H = \sum_{i=0}^{N} H_i \otimes \bar{H}_i \\ &H_0 = E|0\rangle,\ \bar{H}_0 = \bar{E}|0\rangle. \end{aligned}$$

Let P_i ($\bar{P}_i$) be the projector onto the chiral (antichiral) subspace H_i ($\bar{H}_i$). Chiral (antichiral) vertex operators $\varphi^l_{i,k}(z)$ ($\bar{\varphi}^l_{i,k}(z)$) are defined through

$$\varphi^l_{i,k}(z) = P_l \varphi_i(z) P_k \tag{3}$$

$$\bar{\varphi}^l_{i,k}(z) = \bar{P}_l \bar{\varphi}_i(z) \bar{P}_k. \tag{4}$$

We set

$$\Phi_{\mathcal{I}}(z, \bar{z}) = \sum_{l,k} C^l_{ik} \varphi^l_{ik}(z) \otimes \bar{\varphi}^l_{ik}(z) \tag{5}$$

and write formally $\Phi_{\mathcal{I}} = \varphi_i \otimes \bar{\varphi}_i$.

The C^c_{ab} are here expansion coefficients of the operator product algebra (OPA), parametrising the strength of the fusion of operators $\varphi_a \otimes \bar{\varphi}_a$ and $\varphi_b \otimes \bar{\varphi}_b$ into $\varphi_c \otimes \bar{\varphi}_c$. The expansion coefficients are determined through the conformal bootstrap (see below)†.

The above-mentioned conformal blocks appear in the decomposition of correlation functions of physical operators into correlations of chiral and anti-chiral vertex operators:

$$\begin{aligned}
&\langle \Phi_{\mathcal{I}_1}(z_1, \bar{z}_1) \ldots \Phi_{\mathcal{I}_s}(z_s, \bar{z}_s) \rangle \\
&\quad = \sum_{k_1 \ldots k_{s-3}} C^{k_1}_{i_{s-1}, i_s} \left(\prod_{l=2}^{s-3} C^{k_l}_{i_{s-l}, k_{l-1}} \right) C^{i_1}_{i_2, k_{s-3}} \\
&\quad \times \langle \varphi^0_{i_1,i_1}(z_1) \varphi^{i_1}_{i_2,k_{s-3}}(z_2) \ldots \varphi^{k_2}_{i_{s-2},k_1} \varphi^{k_1}_{i_{s-1},i_s} \varphi^{i_s}_{i_s,0}(z_s) \rangle \langle \bar{\varphi}^0_{i_1,i_1} \ldots \bar{\varphi}^{i_s}_{i_s,0} \rangle \qquad s > 3.
\end{aligned} \tag{6}$$

The order of chiral vertex operators chosen on the RHS of (6) is related to a particular fusion scheme (cf figure 1(a)). Other choices of order of vertex operators will be related to different but equivalent fusion schemes. The elementary operations generating the general transformation from one to another fusion scheme are obviously interchanges of neighbouring vertex operators. Such an elementary 'braiding' operation is pictorially represented in figure 1(b). The conformal blocks

$$F^{\boldsymbol{k}}_{\boldsymbol{i}}(z_1, \ldots, z_s) = \langle \varphi^0_{i_1,i_1}(z_1) \ldots \varphi^{k_2}_{i_{s-2},k_1}(z_{s-2}) \varphi^{k_1}_{i_{s-1},i_s}(z_{s-1}) \varphi^{i_s}_{i_s,0}(z_s) \rangle$$

and

$$F^{\boldsymbol{k}'}_{\boldsymbol{i}}(z_1, \ldots, z_s) = \langle \varphi^0_{i_1,i_1}(z_1) \ldots \varphi^{k_2}_{i_{s-1},k'_1}(z_{s-1}) \varphi^{k'_1}_{i_{s-2},i_s}(z_{s-2}) \varphi^{i_s}_{i_s,0}(z_s) \rangle$$

corresponding to figures 1(a) and 1(b) are linearly related:

$$F \ldots {}^{k_2 k_1}_{i_{s-1}, i_{s-2}, i_s}(z_1, \ldots, z_s) = \sum_{k'_1} B^{i_{s-1}, i_{s-2}}_{\pm k_1, k'_1} F' \ldots {}^{k_2, k'_1}_{i_{s-1}, i_{s-2}, i_s}(z_1, \ldots, z_s). \tag{7}$$

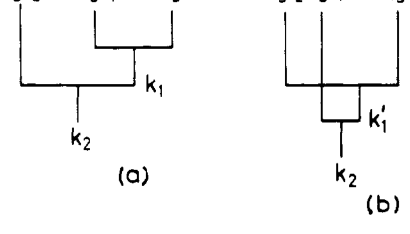

Figure 1.

† We deal here and in the following exclusively with scalar operators. It will be assumed that the fusion algebra is not plagued by degeneracies, i.e. the qualitative fusion constants are supposed to be zero or one.

The block functions F and F' have convergent power series expansions in the sectors

$$R_s = \{(z_1, \ldots, z_s) : |z_s - z_{s-1}| \equiv |z_{s,s-1}| < |z_{s,s-2}| < \ldots < |z_{s,1}|\}$$

and

$$R'_s = \{(z_1, \ldots, z_s) : |z_{s,s-2}| < |z_{s,s-1}| < |z_{s,s-3}| < \ldots\}$$

respectively of the form

$$F_i^k = z_{s,s-1}^{-\Delta^{k_1}_{i_s,i_{s-1}}} z_{s,s-2}^{-\Delta^{k_2}_{k_1,i_{s-2}}} \ldots z_{s,1}^{-\Delta^0_{i_1,i_1}} \sum_{n_i} d_{n_1 \ldots n_{s-1}} \prod_i z_{s,s-i}^{n_i} \tag{8}$$

$$F_i^{k'} = z_{s,s-2}^{-\Delta^{k_1}_{i_s,i_{s-2}}} \; z_{s,s-1}^{-\Delta^{k_2}_{k_1,i_{s-2}}} \; \ldots z_{s,1}^{-\Delta^0_{i_1 i_1}} \sum \ldots \tag{9}$$

$$\Delta^c_{ab} = \Delta_a + \Delta_b - \Delta_c$$

with Δ_a denoting here the scaling dimension of the chiral field φ_a. The coefficients $d_{n_1 \ldots n_{s-1}}$ are fixed through the commutation relations of the symmetry algebra E. The linear relations (7) comprise in fact an analytic continuation of the block functions from R_s to R'_s and vice versa. The analytic continuation can be performed along two homotopically non-equivalent paths giving rise in general to phase ambiguities which are reflected in equation (7) through the labels $\pm$.

The matrices B and their analogues figuring in other permutations of neighbouring vertex operators generate a representation of the braid group. We refer to [10–12] for the ensuing consistency relations. We will need for the following discussion the connection between the expansion coefficients C^a_{bc} (equation (5)) and the braid matrices appearing in equation (7). The important point is that the correlations of physical operators have to be one-valued functions with respect to all variables as they are moving in the Euclidean plane. One-valuedness is guaranteed in the region R because the correlation is given (cf (6)) by a sum of terms each being built up of a holomorphic function and its complex conjugate. The phases arising along paths encircling the origin in the space of difference variables $z_{is} = z_i - z_s$ cancel among the holomorphic and antiholomorphic pieces. The same cancellation mechanism applies also to the correlation functions analytically continued from R_s to R'_s provided that the expansion coefficients satisfy the relations

$$\sum_{k_1} C^{k_2}_{i_{s-2},k_1} C^{k_1}_{i_{s-1},i_s} B_{k_1,k'} B^*_{k_1,k''} = 0 \qquad k' \neq k'' \tag{10}$$

$$\sum_{k_1} C^{k_2}_{i_{s-2},k_1} C^{k_1}_{i_{s-1},i_s} B_{k_1,k'} B^*_{k_1,k'} = C^{k_2}_{i_{s-1},k'} C^{k'}_{i_{s-2},i_s}. \tag{11}$$

Equation (10) fixes in fact the coefficients up to an overall normalisation.

3. Perturbation theory

We are now prepared to enter the discussion of PT around some RCFT, call it R_0, with a perturbation given by equation (1). We will only consider cases in which the scaling dimensions d_{ψ_i} of the interaction $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k)$ satisfy

$$1 < d_{\psi_i} \leqslant 2 \qquad \psi_i = \varphi_i \otimes \bar{\varphi}_i \qquad d_{\psi_i} = \Delta_{\varphi_i} + \Delta_{\bar{\varphi}_i} = 2\Delta_{\varphi_i}. \tag{12}$$

It is assumed that the operators collected in $\boldsymbol{\psi}$ are closed under the OPA modulo irrelevant operators. Indirect couplings to relevant or marginal operators others than those in ψ through multiple fusion processes are also assumed not to exist. One finds through straightforward infrared power counting that the quoted restrictions give rise to an infrared finite PT for all correlation functions†.

Let $X((z_1, \bar{z}_1), \ldots, (z_r, \bar{z}_r))$ denote a bunch of local operators with positions at $(z_1, \bar{z}_1), \ldots, (z_r, \bar{z}_r)$. We are concerned with the Gell-Mann–Low series

$$\langle X((z_1, \bar{z}_1), \ldots, (z_r, \bar{z}_r))\rangle_{S+\Delta S} = \sum_n \frac{1}{n!} \int \mathrm{d}^2x_1 \ldots \mathrm{d}^2x_n \langle g \cdot \psi(x_1) \ldots g \cdot \psi(x_n) X\rangle_S^{(C)} \tag{13}$$

where the disconnected vacuum amplitudes are discarded.

The integrand on the RHS of (13) is given as an $(n+r)$-point correlation function of the chosen conformally invariant theory R_0. An estimate of the singularities occurring at short distances in the Gell-Mann–Low series is provided by the possibility of decomposing the correlations into suitable conformal blocks. Let us inspect the singularities appearing in the nth-order term of the series (13) from a region where all n interaction points are close to each other but are well separated from the positions $(z_1, \bar{z}_1) \ldots (z_r, \bar{z}_r)$ of the external operators. We divide the $2n$-dimensional integration domain into sectors

$$\mathscr{D}_p = \{\boldsymbol{x} = (x_1, \ldots, x_n); |x_{n,p(n-1)}| < |x_{n,p(n-1)}| \ldots < |x_{n,p(1)}|\}$$

with p denoting an element of the permutation group $S_{(n-1)}$. The conformal block functions associated with a fusion scheme being naturally attached to $\mathscr{D}_p$ (cf figure 2) give rise to power-like singularities of the form

$$x_{n,p(n-1)}^{-\Delta^{k_1}_{i_n, i_{p(n-1)}}} \prod_{r=2}^{n-1} x_{n,p(r-1)}^{-\Delta^{-k_r}_{i_{p(n-r)}, k_{r-1}}} \qquad 1 \leqslant i_x \leqslant N. \tag{14}$$

We recall the definition

$$\Delta^c_{ab} = \Delta_a + \Delta_b - \Delta_c$$

with Δ_x being the scaling dimension of the chiral field φ_x. The labels i_x decorating the exponents in (14) are meant to allude to the scaling dimension of the chiral part of the interaction whereas the labels k_x hint at 'intermediate' operators appearing in the fusion scheme according to the rules of the OPA of R_0.

The sum of exponents in (14)

$$\delta^{k_{n-1}} = \Delta_{i_n} + \sum_q \Delta_{i_{p(n-q)}} - \Delta_{k_{n-1}} \tag{15}$$

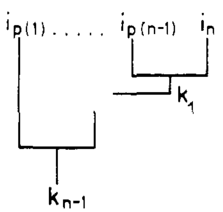

Figure 2.

† At least in the sense of an asymptotic expansion in the deviation from strict marginality.

provides a rough measure for the strength of the singularities present at the coincidence points $x_{n,n-1}=\ldots x_{n,1}=0$ of the correlation function $\langle\psi_{i_1}(x_1,\bar{x}_1)\ldots\psi_{i_n}(x_n,\bar{x}_n)X\rangle$. The singularity is integrable with respect to the integration measure $\Pi_{i=1}^{n-1}\,\mathrm{d}^2x_{n,i}$ in the case

$$2\delta^{k_{n-1}}<2(n-1). \tag{16}$$

(The factor 2 on the LHS of the inequality derives from the fact that singularities of the physical correlations are essentially the modulus squared of those quoted in (14).) Subtractions and therewith the specification of renormalisation constants are necessary in the opposite case

$$2\delta^{k_{n-1}}\geqslant 2(n-1). \tag{17}$$

The conclusion to be drawn from (15)-(17) is that PT around RCFT with marginal or relevant interactions ($2\Delta_i\leqslant 2$) is, with respect to short-distance singularities, not different from standard PT with renormalisable and super-renormalisable interactions around a Gaussian fixed point: the appearance of irrelevant (non-renormalisable) operators, i.e. the occurrence of an operator at the end of the fusion chain with a scaling dimension $\Delta_{k_{n_1}}\geqslant 1$, is not accompanied by a new singularity and irrelevant interactions do not interfere with the renormalisation of the marginal (renormalisable) interactions. (There is no dependence of δ on intermediate operators. The influence of non-renormalisable interactions could only enter through such intermediate operators.)

Our main tool for the analysis of the Gell-Mann-Low (GML) series (equation (13)) consists of a decomposition of multiple two-dimensional integrals over the Euclidean plane into contour integrals through successive application of Stokes' theorem. Hints to the use of Stokes' theorem in this context have been given by Dotsenko in [13] and by one of the present authors in [14].

The nth-order integrand of the GML series decomposes, as discussed in some detail above, into holomorphic and antiholomorphic pieces:

$$\langle\psi_{i_1}(x_1,\bar{x}_1)\ldots\psi_{i_n}(x_n,\bar{x}_n)X\rangle=\sum_{\boldsymbol{k}}\Big(\prod_{k_i}C^{k_1}_{\cdots}\Big)F_i^{\boldsymbol{k}}\bar{F}_i^{\boldsymbol{k}}\qquad \{k_j\}=\boldsymbol{k}.$$

where $F_i^{\boldsymbol{k}}$ denotes a vector of holomorphic conformal block functions (and $\bar{F}$ its complex conjugate). The holomorphy domain of F is given by the universal covering of $\mathbb{C}^{(n+r)}$ with subvarieties of coinciding arguments being removed. We want to look on F for a moment as an analytic function in one variable, say x_1, with the other variables taken as fixed parameters and to restrict it to one Riemann sheet by putting cuts in the complex plane from $x_2\ldots x_n$, $z_1\ldots z_r$ to infinity as indicated in figure 3. The product $F_i\cdot\bar{F}_i$ has clearly no discontinuity over the cuts. A slightly stronger property concerning the absence of discontinuities along the cuts also holds: let $y_{1\pm}$

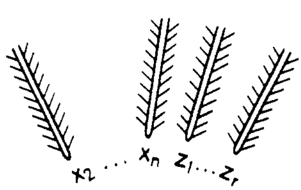

Figure 3.

and $y_{2\pm}$ be two points on some cut with $(\pm)$ denoting boundary values from the RHS and LH respectively. One has the equality

$$F_i^k(y_{1+},\ldots)\bar{F}_i^k(y_{2+},\ldots)=F_i^k(y_{1-},\ldots)\bar{F}_i^k(y_{2-},\ldots). \tag{18}$$

This is a simple consequence of the fact that one might assume according to equations (8) and (9) without loss of generality that the basis of block functions has been chosen so that the components of F_i^k vary over the cut under consideration by constant phase factors.

Let $H_i^k(a)(x_1)$ denote the integral of F_i^k, $H_i^k(a)(x_1)=\int_a^{x_1}\mathrm{d}t\,F_i^k(t;\ldots)$ where the point a is chosen arbitrarily. The integration contour from a to x_1 is taken so that none of the cuts of figure 3 is crossed. The application of Stokes' theorem in the x_1-plane yields

$$\begin{aligned}\int \mathrm{d}^2x_1\, F_i^k\bar{F}_i^k &= \int \mathrm{d}^2x_1\frac{\partial}{\partial x_1}(H_i^k\bar{F}_i^k)\\ &=\frac{1}{2\mathrm{i}}\sum_{l=2}^{n+r}\int_{\bar{\mathscr{C}}_l}\mathrm{d}\bar{x}_1\, H_i^k\bar{F}_i^k\end{aligned} \tag{19}$$

where $\mathscr{C}_l$ denotes a clockwise contour around the cut (cf figure 3) starting at x_l. We use here and in the following the notation $z_i=x_{n+i}$. Decomposing

$$H_i^k(a)(x_1)|_{x_1\in\mathscr{C}_l}=\int_a^{x_l}\mathrm{d}t(\ldots)+\int_{x_l}^{x_1}\mathrm{d}t(\ldots) \tag{20}$$

one finds from equation (18) that only the first term of the RHS in (20) yields a non-vanishing contribution to the contour integral along $\mathscr{C}_l$ in equation (19). We arrive therewith after putting $a=x_2$ at

$$\int \mathrm{d}^2x_1\, F_i^k\bar{F}_i^k=\frac{1}{2\mathrm{i}}\sum_{l=3}^{n+r}\int_{x_2}^{x_l}\mathrm{d}t\,F_i^k\int_{\bar{\mathscr{C}}_l}\mathrm{d}\bar{t}\,\bar{F}_i^k. \tag{21}$$

A more symmetric treatment of the homomorphic and antiholomorphic pieces is still desirable. The contour integral of the holomorphic piece may be rewritten as

$$\int_{x_2}^{x_l}\mathrm{d}t\,F=\left(\int_{\mathscr{C}_{2,-}}\mathrm{d}t\,F+\sum_{2<r<l}\int_{\mathscr{C}_r}\mathrm{d}t\,F+\int_{\mathscr{C}_{l,+}}\mathrm{d}t\,F\right) \tag{22a}$$

$$=-\left(\int_{\mathscr{C}_{2,+}}\mathrm{d}t\,F+\sum_{r>l}\int_{\mathscr{C}_r}\mathrm{d}t\,F+\int_{\mathscr{C}_{l,-}}\mathrm{d}t\,F\right) \tag{22b}$$

where $\mathscr{C}_{x,+}$ and $\mathscr{C}_{x,-}$ denote contours from ∞ to x and from x to ∞ respectively such that

$$\mathscr{C}_x=\mathscr{C}_{x,+}+\mathscr{C}_{x,-} \tag{23}$$

holds.

We can freely move from one system of conformal block functions to another one (cf equations (7), (10) and (11)) in order to relate the integrals $\int_{\mathscr{C}_{l,\pm}}$ to $\int_{\mathscr{C}_l}$. Let $F_i^{k(1,l)}$ be a basis of block functions associated with a fusion scheme in which the operators $\varphi_{i_1}(x_1)$ and $\varphi_{i_l}(x_l)$ are coupled in the first fusion step to operators which we denote by $\varphi^k(x_l)$. The vector components of $F_i^{k(1,l)}$ are (partially) classified through the intermediate operators $\varphi^k(x_l)$:

$$F_i^{k(1,l)}=F_{\ldots i_1 i_l}^{\ldots k(1,l)}.$$

The component $F^{\dots k(1,l)}_{\dots i_1 i_l}$ picks up along the contour $\mathscr{C}_{l,-}$, a phase factor $\exp[2\pi i(\Delta_{i_1}+\Delta_{i_l}-\Delta_k)]\equiv 2\pi i\Delta^k_{i_1,i_l}$ relative to the position on the contour $\mathscr{C}_{l,+}$. This leads together with (23) to the relations

$$\int_{\mathscr{C}_{l,+}} F^{\dots k(1,l)}_{i} = (1-\mathrm{e}^{2\pi\mathrm{i}\Delta^k_{i_1,i_l}})^{-1}\int_{\mathscr{C}_l} F^{\dots k(1,l)}_{i} \tag{24}$$

$$\int_{\mathscr{C}_{l,-}} = (1-\mathrm{e}^{-2\pi\mathrm{i}\Delta^k_{i_1,i_l}})^{-1}\int_{\mathscr{C}_l} \dots . \tag{25}$$

Inserting (24) and (25) into the half the sum of (22a) and (22b) and going back with this to (21) we obtain

$$\sum_k \left(\prod C\right)\int \mathrm{d}^2x_1\, F^k_i \bar{F}^k_i$$
$$= \frac{1}{4}\sum_k \sum_{j,l\geq 2}\left(\prod C\right)\int_{\mathscr{C}_l} F^{k(1,j)}_i\,\mathrm{d}t \int_{\bar{\mathscr{C}}_l} \bar{F}^{k(1,j)}_i\,\mathrm{d}\bar{t}$$
$$\times\left(\delta^{jl}\frac{\cos\pi\Delta^k_{i_1,i_l}}{\sin\pi\Delta k^k_{i_1,i_l}}+\frac{1}{\mathrm{i}}\,\varepsilon(l-j)\big|_{j,l\geq 3}\right) \tag{26}$$

where

$$\varepsilon(x)=\begin{cases} 0 & x=0\\ 1 & x>0\\ -1 & x<0\end{cases}$$

and $\Pi\, C$ stands for the product of the coefficients C which depend on $\boldsymbol{k}$ corresponding to the particular fusion scheme which has been used. We have therewith completed the first step in our programme to rewrite two-dimensional integrals over the Euclidean plane as pairs of contour integrals. Let us assume now that k integrations have already been treated in this spirit. We will make use of the abbreviating notation

$$\underbrace{\int_{\mathscr{C}_{l_1}}\mathrm{d}t_{i_1}\dots\int_{\mathscr{C}_{l_1}}\mathrm{d}t_{i_{s_1}}}_{s_1\ \text{contours}}\dots\underbrace{\int_{\mathscr{C}_{l_r}}\mathrm{d}t_{i_{N+1}}\dots\int_{\mathscr{C}_{l_r}}\mathrm{d}t_{i_{N+s_r}}}_{s_r\ \text{contours}} F^{\boldsymbol{k}}_i=\mathscr{C}_{l_1^{s_1}\dots l_r^{s_r};i}{}^{\boldsymbol{k}}\equiv\mathscr{C}^{\boldsymbol{k}}_{\boldsymbol{s};i}$$

and make use of an analogous notation $\bar{\mathscr{C}}^{\boldsymbol{k}}_{\bar{\boldsymbol{s}},i}$ for the complex conjugate integrals. Contour $\mathscr{C}_l$ is meant as before to swing around a cut starting at x_l. Multiple contours ($s_l>1$) around the same cut are taken such that one contour is enclosed by another without crossings (cf figure 4). Our assumption, strictly speaking an induction hypothesis, amounts to a relation of the type:

$$\int \mathrm{d}^2t_1\dots\mathrm{d}^2t_k\, F^{\boldsymbol{k}}_i\bar{F}^{\boldsymbol{k}}_i=\sum_{\substack{|\boldsymbol{s}|=|\bar{\boldsymbol{s}}|=k\\ i',\boldsymbol{k}'}} h^{\boldsymbol{k}'}_{\boldsymbol{s},\bar{\boldsymbol{s}};i'}\mathscr{C}^{\boldsymbol{k}'}_{\boldsymbol{s};i'}\bar{\mathscr{C}}^{\boldsymbol{k}'}_{\bar{\boldsymbol{s}};i'} \qquad |\overset{(-)}{\boldsymbol{s}}|=\sum \overset{(-)}{s_l} \tag{27}$$

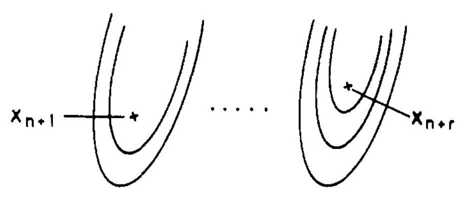

Figure 4.

with coefficients $h_{s,\bar{s}}$ being determined through braid matrices and the spectrum of scaling dimensions of the RCFT R_0. This will be explained further below. Note that $\bar{\mathscr{C}}_{\bar{s}}$ is the complex conjugate of $\mathscr{C}_{\bar{s}}$.

We perform the transformation of the integral $\int d^2 t_{k+1}$ in order to verify the general validity of relations like equation (27) and to obtain at the same time insight into the structure of the coefficients $h_{s,\bar{s}}$. We may assume without loss of generality that terms with contours around x_{k+1} do not appear on the RHS of equation (27). This can always been achieved through application of Cauchy's theorem to $\mathscr{C}_{l_1^{s_1-1} l_2^{s_2} \dots l_m^{s_m}}$ which with the present notation reads as

$$\mathscr{C}_{l_1^{s_1} \dots l_m^{s_m}} - \mathscr{C}_{l_1^{s_1-1} l_2^{s_2+1} \dots l_m^{s_m}} - \dots - \mathscr{C}_{l_1^{s_1-1} \dots l_m^{s_m+1}} = 0. \tag{28}$$

The manipulation which led us from equation (19) to equation (21) can easily be adapted.

$$\sum_{|s|=|\bar{s}|=k} h_{s,\bar{s}} \int d^2 t_{k+1}\, \mathscr{C}_s \bar{\mathscr{C}}_{\bar{s}}$$

$$= \frac{1}{4i} \sum_{\substack{l \geqslant k+2 \\ m \geqslant k+2}} \sum_{|s|=|\bar{s}|=k} h_{s,\bar{s}} \bar{\mathscr{C}}_{\bar{s},m}$$

$$\times ((\mathscr{C}_{s,l+} - \mathscr{C}_{s,l-})\delta_{l,m} + \varepsilon(l-m)\mathscr{C}_{s,l}|_{l,m \geqslant k+3}). \tag{29}$$

The dependence on block $(\boldsymbol{i}, \boldsymbol{k})$ labels has been suppressed here in $\mathscr{C}_{s,\dots}$ $(\bar{\mathscr{C}}_{\bar{s},\dots})$. The contour of t_{k+1} in $\mathscr{C}_{s,l+}$ $(\mathscr{C}_{s,l-})$ is supposed to enclose all other contours at the same position x_l. We will have achieved our goal to recover from the RHS of (29) a representation analogous to the RHS of equation (27) if we can replace the terms $(\mathscr{C}_{s,l+} - \mathscr{C}_{s,l-})$ on the RHS of equation (29) by 'complete' contour integrals (cf figure 4). Let us inspect closer as a representative example (the other integrals have to be treated similarly) the term

$$\mathscr{C}_{l^k,l+},\ l > k+1.$$

The first k contours encircle here x_l and the contour of t_{k+1} goes to the left of the contours of $t_1, \dots, t_k$. Let $\mathscr{D}_l^{(k+1)}$ denote the union of contours of $t_1, \dots, t_{k+1}$ which build up $\mathscr{C}_{l^k,l+}$.

The following decomposition of $\mathscr{C}_{l^k,l+}$ will be useful:

$$\mathscr{C}_{l^k,l+} = \sum_{P \in S_{k+1}} \mathscr{C}^P_{l^k,l+}$$

$$\mathscr{C}^P_{l^k,l+} = \int dt_1 \dots dt_{k+1}(\dots),\ (t_1, \dots, t_{k+1}) \in \mathscr{D}_l^{(k+1),P}$$

$$\mathscr{D}_l^{(k+1),P} = \{(t_1, \dots, t_{k+1}) \in \mathscr{D}_l^{(k+1)};\ |t_{l,P(k+1)}| > \dots > |t_{l,P(1)}|\}.$$

S_{k+1} denotes here the permutation group of $k+1$ elements and $t_{l,m} = x_l - t_m$. We choose for every sector $\mathscr{D}_l^{(k+1),P}$ a basis of conformal blocks in which the singularities with respect to the difference variables $t_{l,P(i)}$, $i = 1, \dots, k+1$, appear in diagonalised form (the necessity to change from one set of blocks to another one induces here—and similarly at other places—a dependence on braid matrices). A typical contribution to a component $\mathscr{C}_{l^k,l+}$ is of the form

$$\prod (1 - e^{2\pi i(\dots)}) \int_{\infty}^{x_{l+}} dt_{P(k+1)} \int_{t_{P(k+1)}}^{x_{l-}} dt_{P(k)} \dots \int_{t_{P(2)}}^{x_{l+}} dt_{P(1)}. \tag{30}$$

We have chosen contours on the LHS of the cut starting at x_l instead of contours on both sides. The prefactor $\Pi\,(1-e^{2\pi i(\ldots)})$ takes care of the difference between the two choices. (The very point of decomposing $\mathscr{C}_{l^k,l+}$ into pieces $\mathscr{C}^P_{l^k,l+}$ and of choosing in the sectors $\mathscr{C}_l^{(k+1),P}$ appropriate bases of conformal blocks is connected with the fact that one can find only in this way a separation of contributions whose discontinuities over the cut are governed by constant phase factors.) We can further transform the RHS of (30) by the following self-evident manipulations†:

$$\int_{\infty}^{x_{l+}} \mathrm{d}t_{P(k+1)} \int_{t_{P(k+1)}}^{x_{l+}} \ldots \int_{t_{P(2)}}^{x_{l+}} \mathrm{d}t_{P(1)}(\ldots)$$

$$= \sum_F \left(1-\exp 2\pi \mathrm{i}\left(\sum_{r=1}^{k+1} \Delta_{i_r} - \Delta_F\right)\right)^{-1}$$

$$\times \int_{\mathscr{C}_l} \mathrm{d}t_{P(k+1)} \int_{x_{P(k+1)-}}^{x_l} \mathrm{d}t_{P(k)} \ldots (\ldots) \tag{31a}$$

$$= \sum_F (1-\exp 2\pi\mathrm{i}(\ldots))^{-1} \sum_s \int_{\mathscr{C}_l} \mathrm{d}t_{P(k+1)} \int_{t_{P(k+1)-}}^{\infty} \mathrm{d}t_{P(k)} \ldots \int_{t_{P(k+1-s)-}}^{\infty} \mathrm{d}t_{P(k-s)}$$

$$\times \int_{\infty}^{x_{l+}} \mathrm{d}t_{P(k-s-1)} \ldots \int_{t_{P(2)}}^{x_{l-}} \mathrm{d}t_{P(1)}(\ldots). \tag{31b}$$

We have managed in (31b) to rewrite (30) such that only less than $k+1$ contours appear at a time either along the cut from $t_{P(k+1)}$ to ∞ or along the one starting at x_l.

The problem of re-expressing these integrals (with less than $(k+1)$ contours) through 'complete' integrals has supposedly been solved in previous induction steps. One has to make iterative use of identities of the kind

$$\int_{\mathscr{C}_l} \mathrm{d}t_x \int_{\mathscr{C}_x} \mathrm{d}t_y(\ldots) = \int_{\mathscr{C}_l} \mathrm{d}t_y \int_{\mathscr{C}_l} \mathrm{d}t_x - \int_{\mathscr{C}_l} \mathrm{d}t_x \int_{\mathscr{C}_l} \mathrm{d}t_y$$

(with the understanding that on the RHS the contour of the integral standing to the left encloses the contour of the integral on the right) to arrive finally at a representation as in (27). We will not give in this paper the explicit expressions for the general nth-order term of the GML series but confine ourselves to describe in the next section some details of the third-order term. We end this section with the following concluding remarks.

(a) The integrals on the RHS of (29) are free from ultraviolet (UV) singularities since the contours can be deformed such that the interaction points never approach each other beyond some finite distance. Infrared divergences of the integrals are excluded (after the subtraction of disconnected vacuum amplitudes) through the assumptions made at the beginning of this section.

(b) The possible UV singularities are encoded into the (hypothetical) meromorphic behaviour of the coefficients $h_{s,\bar{s}}$ with respect to the spectrum of operators in R_0.

(c) Looking back on the above manipulations which were designed in order to transform the $(k+1)$th integration, one easily realises the general structure of the coefficients $h_{s,\bar{s}}$. It is given by products of braid matrix elements and ratios of trigonometric functions, the arguments of which are proportional to combinations of scaling dimensions.

† In (31a) Δ_F denotes the scaling dimension of the operator appearing at the end of the fusion scheme related to the sector $\mathscr{D}_l^{(k+1),P}$.

4. Coupling constant renormalisation to third order

We want to discuss some of the details of the calculations to evaluate the third order in the coupling constant. To fix the ideas let us think of the unitary series of RCFT with central Virasoro charge

$$c_m = 1 - \frac{6}{m(m+1)} \qquad m \in \mathbb{Z} \qquad m \geqslant 3$$

(see Friedan *et al* quoted in [3]). A closed subalgebra of the OPA of the models is built up by scalar operators denoted

$$\psi_{(1,s),(1,s)} = \varphi_{1,s} \otimes \bar{\varphi}_{1,s} \qquad s = 2, 3, \ldots$$

with scaling dimensions

$$d^{(m)}_{1,s} = 2\Delta^{(m)}_{1,s} = ((m(s-1)-1)^2 - 1)\frac{1}{2m(m+1)}. \tag{32}$$

We consider, following Cardy and Ludwig [6], the perturbation

$$\Delta S = g \int \mathrm{d}^2 x\, \psi_{(1,3),(1,3)}. \tag{33}$$

Operators figuring according to the OPA in the GML series of ΔS are the unit operator, $\psi_{(1,3),(1,3)}$ itself with scaling dimension

$$d^{(m)}_{1,3} = 2\Delta^{(m)}_{1,3} \equiv 2 - y_m = 2 - \frac{4}{m} + \mathrm{O}\left(\frac{1}{m^2}\right) \tag{34}$$

and irrelevant operators. The contributions coming from fusions ending up with the unit operator are discarded since these amount to contributions of disconnected vacuum amplitudes. The particular simplicity of the chosen perturbation is due to the fact that no other relevant operators are induced through fusions, than just the one we started from. It means that one has to handle only a single coupling constant.

The formulae for the GML series up to second order in the coupling constant have already been worked out in the previous section. We repeat these formulae here for convenience. Let $\psi_i = \varphi_i \otimes \bar{\varphi}_i$ be some arbitrary local operators of the RCFT under consideration. Setting $\varphi_{1,3} \equiv \varphi$ and $\bar{\varphi}_{1,3} \equiv \bar{\varphi}$ we have

$$\begin{aligned}
&\int \mathrm{d}^2 x_1 \langle \varphi(x_1) \otimes \bar{\varphi}(\bar{x}_1) \varphi(x_2) \otimes \bar{\varphi}(\bar{x}_2) \psi_1(x_3, \bar{x}_3) \ldots \psi_r(x_{r+2}, \bar{x}_{r+2}) \rangle \\
&\qquad = \sum_k \sum_{j,l \geqslant 2} \left(\prod C\right) \frac{1}{4} \int_{\mathscr{C}_j} \mathrm{d}x_1\, F^k_{(1,j)}(x_1, \ldots) \int_{\bar{\mathscr{C}}_l} \mathrm{d}\bar{x}_1\, \bar{F}^k_{(1,j)}(\bar{x}_1, \ldots) \\
&\qquad \times \left(\delta_{l_j} \cos(\pi \Delta^k_{(1,3),(1,3)}) \sin^{-1}(\pi \Delta^k_{(1,3),(1,3)}) + \frac{1}{\mathrm{i}} \varepsilon(l-j)|_{j,l \geqslant 3} \right).
\end{aligned} \tag{35}$$

The subscripts $(1, l)$ of the block functions on the RHS refer to a base in which the operators $\overset{(-)}{\varphi}(x_1)$ and $\overset{(-)}{\varphi}_l(x_l)$ are fused at the first place whereas the superscript k refers to the particular operator emerging from that first fusion.

Our main purpose consists in displaying the above mentioned meromorphic structure in the scaling dimension and to verify that the subtraction of pole terms can be interpreted as addition of local counterterms (in the spirit of field theoretical Lagrangean PT). We concentrate on the pieces

$$\sum_k \left(\prod C\right) \int_{\mathscr{C}_2} \mathrm{d}x_1\, F^k_{(1,2)} \int_{\bar{\mathscr{C}}_2} \mathrm{d}\bar{x}_1\, \bar{F}^k_{(1,2)}(\ldots) \frac{\cos \pi\Delta^k_{1,3}}{\sin \pi\Delta^k_{1,3}} \tag{36}$$

from the RHS of equation (35) whose singularity structure can be related to renormalisation of the coupling constant. (Singularities going along with contour integrals around $\overset{(-)}{x}_3, \ldots, \overset{(-)}{x}_{3+r}$ are related to wavefunction renormalisation.) The first term in the sum (36) is supposed to represent the contribution of that fusion which reproduced $\varphi(\bar{\varphi})$.

The following terms correspond to couplings to irrelevant operators. The prefactor of the first term has a simple pole for $\psi_{(1,3)(1,3)}$ approaching marginality ($\Delta^{(m)}_{1,3} \to 1 \Leftrightarrow y_m \to 0$):

$$\frac{1}{4} \frac{\cos \pi\Delta_{1,3}}{\sin \pi\Delta_{1,3}} = \frac{1}{4\pi y_m} + \mathrm{O}(y^0_m). \tag{37}$$

From both contour integrals $\int_{\mathscr{C}_2} \mathrm{d}x_1$ and $\int_{\bar{\mathscr{C}}_2} \mathrm{d}\bar{x}_1$ there survive in this limit only the residua of the (simple) pole term of the integrand in the complex difference variables $x_{1,2}$ and $\bar{x}_{1,2}$ respectively. The non-local parts of the integrals drop out since the discontinuities over the cut vanish in this limit. The result is:

$$\begin{aligned} &\int \mathrm{d}^2x_1 \langle \psi_{1,3}(x_1, \bar{x}_1) \psi_{1,3}(x_2, \bar{x}_2) \prod_i \varphi_i \otimes \bar{\varphi}_i(x_{3+i}, \bar{x}_{3+i}) \rangle \\ &\qquad = -\pi C^{(1,3)}_{(1,3),(1,3)} \frac{1}{y_m} \langle \psi_{1,3}(x_2, \bar{x}_2) \ldots \rangle + \mathrm{O}(y^0_m). \end{aligned} \tag{38}$$

It might happen that some of the prefactors of the other terms in (35) become accidentally singular (if one hits an integer $\Delta^k_{(1,3)(1,3)}$). But these singularities are compensated through the simultaneously vanishing of the discontinuity over the cut. There are in this case (where irrelevant operators emerge from the fusion process) no non-smooth terms at the coincidence point $x_1 = x_2$ and, $\bar{x}_1 = \bar{x}_2$.

Let us now turn to the partial evaluation of a third-order term of the GML series:

$$\begin{aligned} &\int \mathrm{d}^2x_1\, \mathrm{d}^2x_2 \langle \varphi\otimes\bar{\varphi}(x_1, \bar{x}_1) \ldots \varphi\otimes\bar{\varphi}(x_3, \bar{x}_3) \psi_1(x_4, \bar{x}_4) \ldots \psi_n(x_{n+3}, \bar{x}_{n+3}) \rangle \\ &\qquad = \int \mathrm{d}^2x_2 \sum_k \left(\prod C\right) \sum_{l,j \geqslant 2} A^k_{lj} \int_{\mathscr{C}_l} \mathrm{d}t_1\, F^k_{(1,l)} \int_{\bar{\mathscr{C}}_j} \mathrm{d}\bar{t}_1\, \bar{F}^k_{(1,l)} \qquad (39) \\ &\qquad = \frac{1}{2i} \sum_{s>3} \left(\int_{\mathscr{C}_{3,-}} \mathrm{d}t_2 + \sum_{3<r<s} \int_{\mathscr{C}_r} \mathrm{d}t_2 + \int_{\mathscr{C}_{s,+}} \mathrm{d}t_2 \right) \int_{\bar{\mathscr{C}}_s} \mathrm{d}\bar{t}_2 \\ &\qquad \times \sum_k \sum_{l,j} \left(\prod C\right) A^k_{lj} \int_{\mathscr{C}_l} \mathrm{d}t_1 \ldots \int_{\bar{\mathscr{C}}_j} \mathrm{d}\bar{t}_1 \ldots \qquad (40) \\ &\qquad A^k_{lj} = \frac{1}{4} \frac{\cos \pi\Delta^k_{lj}}{\sin \pi\Delta^k_{lj}} \delta_{lj} + \frac{1}{4\mathrm{i}} \varepsilon(j-l). \end{aligned}$$

The first equality is obtained through insertion of the previous calculation (equation (35)). The second equality results from the decomposition of the integral $\int \mathrm{d}^3x_2$ over the plane into contour integrals $\int \mathrm{d}t_2$ and $\int \mathrm{d}\bar{t}_2$ respectively. (We repeat here the calculations which lead us above to equation (21).) We content ourselves to discuss the singularities of that part of the integrals in (40) where both arguments t_1 and t_2 ($\bar{t}_1$ and $\bar{t}_2$) are near to x_3 ($\bar{x}_3$). We intend to extract the coefficient of the second-order pole in y_m. The pieces of (40) contributing potentially to the pole term are those where the contours of t_2 and $\bar{t}_2$ are along the cut starting at x_3 and $\bar{x}_3$ respectively and where at least one of the contours in $t_1(\bar{t}_1)$ surround x_2 ($\bar{x}_2$) or x_3 ($\bar{x}_3$) or both points. The terms in question are of the form

$$A=-\frac{1}{16}\int_{\mathscr{C}_{3,+}-\mathscr{C}_{3,-}} \mathrm{d}t_2 \int_{\bar{\mathscr{C}}_3} \mathrm{d}\bar{t}_2 \left\{ \int_{\mathscr{C}_2+\mathscr{C}_3} \mathrm{d}t_1 \int_{\bar{\mathscr{C}}_2+\bar{\mathscr{C}}_3} \mathrm{d}\bar{t}_1 + 2\int_{\mathscr{C}_{2,-}+\mathscr{C}_{3,+}} \mathrm{d}t_1 \int_{\bar{\mathscr{C}}_3} \mathrm{d}\bar{t}_1 - 2\int_{\mathscr{C}_{2,-}+\mathscr{C}_3} \mathrm{d}t_1 \int_{\bar{\mathscr{C}}_2+\bar{\mathscr{C}}_3} \mathrm{d}\bar{t}_1 \right\} \tag{41}$$

where we use the notation

$$\int_{\mathscr{C}_a\pm\mathscr{C}_b} = \int_{\mathscr{C}_a} \pm \int_{\mathscr{C}_b}.$$

The t_2 ($\bar{t}_2$) contour in (41) is supposed to be enclosed by the t_1 ($\bar{t}_1$) contour if the latter extends over $\mathscr{C}_2+\mathscr{C}_3$ ($\bar{\mathscr{C}}_2+\bar{\mathscr{C}}_3$). The t_1 contour is otherwise put 'nearer' to the cut starting at x_3 ($\bar{x}_3$) than the t_2 ($\bar{t}_2$) contour. Following the general strategy explained above (cf equations (28)–(31)) we first express A in terms of integrals which are path-ordered along the x_3-cut. Let us use the definitions

$$\boldsymbol{X}^{(1)} = \int_{\mathscr{C}_3} \mathrm{d}t_2 \int_{t_2}^{x_3} \mathrm{d}t_1 (\ldots)_{\underbrace{213}} \tag{42}$$

$$\boldsymbol{X}^{(2)} = \int_{\mathscr{C}_3} \mathrm{d}t_1 \int_{t_1}^{x_3} \mathrm{d}t_2 (\ldots)_{\underbrace{123}} \tag{43}$$

The indices $\underbrace{213}$ and $\underbrace{123}$ refer to the bases of conformal blocks

$$\langle \ldots \varphi^{\beta}_{\varphi\alpha}(t_2)\varphi^{\alpha}_{\varphi\varphi}(t_1)\varphi^{\varphi}_{\varphi 0}(x_3)\rangle \tag{44}$$

and

$$\langle \ldots \varphi^{\beta}_{\varphi\alpha}(t_1)\varphi^{\alpha}_{\varphi\varphi}(t_2)\varphi^{\varphi}_{\varphi 0}(x_3)\rangle \tag{45}$$

respectively. We recall here the notation (cf equation (3))

$$\varphi^{\beta}_{\varphi\alpha} = P_\beta \varphi P_\alpha .$$

The bases (44) and (45) are related to each other through braiding matrices as follows:

$$\sum_{\alpha'} (B_\pm)^{\alpha'}_{\alpha} \langle \ldots \varphi^{\beta}_{\varphi\alpha'}(t_2)\varphi^{\alpha'}_{\varphi\varphi}(t_1)\varphi^{\varphi}_{\varphi 0}(x_3)\rangle = \langle \ldots \varphi^{\beta}_{\varphi\alpha}(t_1)\varphi^{\alpha}_{\varphi\varphi}(t_2)\varphi^{\varphi}_{\varphi 0}(x_3)\rangle \tag{46}$$

$$\sum_{\alpha'} (B_\pm)^{\alpha'}_{\alpha} \langle \ldots \varphi^{\beta}_{\varphi\alpha'}(t_1)\varphi^{\alpha'}_{\varphi\varphi}(t_2)\varphi^{\varphi}_{\varphi 0}(x_3)\rangle = \langle \ldots \varphi^{\beta}_{\varphi\alpha}(t_2)\varphi^{\alpha}_{\varphi\varphi}(t_1)\varphi^{\varphi}_{\varphi 0}(x_3)\rangle . \tag{47}$$

Here the dependence of the braiding matrices on the block labels β and φ is suppressed. We will also use the notations

$$(P_1)_{\alpha\alpha'} = \delta_{\alpha\alpha'}\, \mathrm{e}^{2\pi i \Delta^{\alpha}_{\varphi\varphi}}$$

$$(P_2)_{\alpha\alpha'} = \delta_{\alpha\alpha'}\, \mathrm{e}^{2\pi i \Delta^{\beta}_{\varphi\alpha}} .$$

Rewriting (41) in terms of the functions $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ and their complex conjugates one obtains

$$A = -\frac{1}{16} \frac{1}{|1-P_1P_2|^2} \hat{\hat{X}} \mathcal{M} \hat{X} \tag{48}$$

where

$$\hat{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix} \qquad \hat{\hat{X}} = (\boldsymbol{X}_1^*, \boldsymbol{X}_2^*) \qquad \mathcal{M} = \begin{pmatrix} \mathcal{M}_{11}\mathcal{M}_{21} \\ \mathcal{M}_{12}\mathcal{M}_{22} \end{pmatrix}$$

$$\mathcal{M}_{11} = (1+P_1P_2)(1-P_1^*)(1-P_2^*) - (1-P_1^*)B_+^\dagger(1-P_1^*)(1-P_1)B_+(1-P_1)$$

$$\mathcal{M}_{22} = (1+P_1)(1+P_2)(1-P_1^*)1-P_2^*)$$

$$\mathcal{M}_{21} = \mathcal{M}_{12}^\dagger = -(1-P_1)(1-P_1^*)(1-P_2^*)B_+(1-P) + 2(1-P_1^*)B_+^\dagger(1-P_1^*)(1+P_1P_2).$$

Our final goal is to express A in terms of the following set of functions which are represented through 'complete' contours:

$$\boldsymbol{Y}_1 = \int_{\mathscr{C}_1+\mathscr{C}_3} \mathrm{d}t_2 \int_{\mathscr{C}_3} \mathrm{d}t_1(\ldots)_{\underset{\smile}{213}} \tag{49}$$

$$\boldsymbol{Y}_2 = \int_{\mathscr{C}_2+\mathscr{C}_3} \mathrm{d}t_1 \int_{\mathscr{C}_3} \mathrm{d}t_2(\ldots)_{\underset{\smile}{123}}. \tag{50}$$

The reversed relations read as

$$\boldsymbol{X}_1 = \frac{1}{1-P_1} Y_1 + D^{-1} \frac{1}{1-P_1^*} D(B_- Y_2 - Y_1) \tag{51}$$

$$\boldsymbol{X}_2 = \frac{1}{1-P_2} Y_2 + D^{-1} \frac{1}{1-P_1^*} D(B_- Y_1 - Y_2) \tag{52}$$

where

$$D = (P_1^*)^{1/2} B_- (P_1^*)^{1/2}.$$

The verification of the fact that (49), (50) and (51), (52) are indeed the inverse of each other is achieved through the above-mentioned polynomial consistency equation. Inserting equations (51), (52)) in equation (48) and including C we arrive at

$$A + \frac{1}{16} \hat{\hat{Y}} \cdot C\hat{Y} = \frac{2}{16} \hat{\hat{Y}} \frac{(1+P_1P_2)}{(1-P_1P_2)} T\hat{Y} \tag{53}$$

where

$$\hat{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \qquad T = \begin{pmatrix} T_{11} T_{12} \\ T_{21} T_{22} \end{pmatrix}$$

$$T_{11} = T_{22} = \frac{C}{1-P_1^*} - D^+ \frac{C}{1-P_1} D$$

$$T_{12} = B_-^\dagger D^\dagger \frac{1}{1-P_1^*} D.$$

The diagonal matrix C comprises here the dependence on the coefficients of the operator product algebra. The leading singularities for y_m near to one are easily extracted from equation (53) if one takes into account the relations

$$\begin{aligned} B_-^{+} D^{+} &= D^{+}(P_1^*)^{1/2} \\ (P_1^*)^{1/2}_{\varphi\varphi} &= -1 + \mathrm{O}(y_m) \end{aligned} \tag{54}$$

and exploits the fact that the combinations of functions

$$Z_1 = D(B_- Y_2 - Y_1)$$

$$Z_2 = D(B_- Y_1 - Y_2)$$

are to be identified with the following 'complete' contour integral expressions

$$Z_1 = \int_{\mathscr{C}_2} \mathrm{d}t_1 \int_{\mathscr{C}_3} \mathrm{d}t_2 (\ldots)_{\substack{312 \\ \omega}} \tag{55}$$

$$Z_2 = \int_{\mathscr{C}_1} \mathrm{d}t_2 \int_{\mathscr{C}_3} \mathrm{d}t_1 (\ldots)_{\substack{321 \\ \omega}} . \tag{56}$$

The result is

$$A = \frac{3\pi^2}{8} \frac{(C_{\varphi\varphi}^{\varphi})^2}{y_m^2} \langle \ldots \varphi(x_3) \rangle + \mathrm{O}\left(\frac{1}{y_m}\right)$$

which could have been guessed in advance. Our purpose here was to show explicitly the locality of the leading singularity. (Locality is no longer manifest if one decomposes the volume integrals of the GML series into contour integrals.) The main ingredient for this demonstration is the arsenal of polynomial consistency relations which are underlying equations (51), (52), (54), (55) and (56).

5. Outlook

Our main purpose in this paper was to show that PT around RCFT with relevant perturbations can be organised such that the structure of the whole series is governed through data available from four point correlations which are braid matrices and the spectrum of singular exponents of the primary local operators. We believe that it is comparatively easy within this framework to establish convergence. We will come back to this point elsewhere.

Acknowledgments

FC thanks IHES and K Gawedzki for hospitality and the Volkswagen-Stiftung for financial support. RF acknowledges financial support of the CNRS and thanks the members of the LPTHE for hospitality. He is indebted to P K Mitter for many encouraging discussions.

References

[1] Polyakov A M 1974 *Sov. Phys.-JETP* **39** 10
Mack G 1976 *New Developments in Quantum Field Theory and Statistical Mechanics* ed M Levy and P K Mitter (New York: Plenum) and references therein
[2] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
[3] Friedan D, Qiu Z and Shenker S H 1984 *Phys. Rev. Lett.* **52** 1575
Bershadsky M, Knizhnik V and Teitelman M 1985 *Phys. Lett.* **151B** 31
Boucher W, Friedan D and Kent A 1986 *Phys. Lett.* **172B** 316
di Vecchia P, Petersen J L and Zheng H B 1986 *Phys. Lett.* **174B** 280
Nam S 1986 *Phys. Lett.* **172B** 323
Zamolodchikov A B and Fateev A **62** *Sov. Phys.-JETP* 215
Knizhnik V G and Zamolodchikov A 1984 *Nucl. Phys.* **247** 83
[4] Zamolodchikov A B 1988 *JETP Lett.* **43** 730
[5] Ludwig A 1987 *Nucl. Phys.* B **285** (FS 19) 97
[6] Cardy J and Ludwig A 1987 *Nucl. Phys.* B **285** (FS 19) 687
[7] Kastor D A, Martinec E J and Shenker S H *Preprint* EFJ-88-31
[8] Zamolodchikov A B 1987 *Zh. Eksp. Teor. Fiz. Pis. Red.* **46** 129
[9] Tsuchiya A and Kanie Y 1987 *Lett. Math. Phys.* **13** 303
[10] Froehlich J 1987 *Nonperturbative Quantum Field Theory* (New York: Plenum) in press
[11] Rehren K H and Schroer B 1988 *Nucl. Phys.* B **295** 229
Rehren K H 1988 *Commun. Math. Phys.* **116** 675
[12] Moore G and Seiberg N 1988 *Phys. Lett.* **122B** 451
[13] Dotsenko Vl S 1988 *Adv. Stud. Pure Math.* **16** 17
[14] Flume R 1988 *Conformal Invariance and String Theory* ed P Dita and V Georgescu (New York: Academic)